

Circular Backbone Colorings: on matching and tree backbones of planar graphs

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Abstract. Given a graph G , and a spanning subgraph H of G , a circular q -backbone k -coloring of (G, H) is a proper k -coloring c of G such that $q \leq |c(u) - c(v)| \leq k - q$, for every edge $uv \in E(H)$. The circular q -backbone chromatic number of (G, H) , denoted by $CBC_q(G, H)$, is the minimum integer k for which there exists a circular q -backbone k -coloring of (G, H) . The Four Color Theorem implies that whenever G is planar, we have $CBC_2(G, H) \leq 8$. It is conjectured that this upper bound can be improved to 7 when H is a tree, and to 6 when H is a matching. In this work, we show that: 1) if G is planar and has no C_4 as subgraph, and H is a linear spanning forest of G , then $CBC_2(G, H) \leq 7$; 2) if G is a plane graph having no two 3-faces sharing an edge, and H is a matching of G , then $CBC_2(G, H) \leq 6$; and 3) if G is planar and has no C_4 nor C_5 as subgraph, and H is a matching of G , then $CBC_2(G, H) \leq 5$. These results partially answers questions posed by Broersma, Fujisawa and Yoshimoto (2003), and by Broersma, Fomin and Golovach (2007). It also points towards a positive answer for the Steinberg's Conjecture.

Keywords: graph coloring, circular backbone coloring, matching, planar graph, Steinberg's conjecture.

1 Introduction

For basic notions and terminology on Graph Theory, the reader is referred to [5]. In this text, we only consider simple graphs.

Let $G = (V, E)$ be a graph. A (*proper*) k -coloring of G is a function $c : V(G) \rightarrow \{1, \dots, k\}$ such that $c(u) \neq c(v)$, for every edge $uv \in E(G)$. G is k -colorable if there exists a k -coloring of G . The *chromatic number* of G , denoted by $\chi(G)$, is the smallest k for which G has a k -coloring. G is k -chromatic if $\chi(G) = k$. The VERTEX COLORING PROBLEM consists in determining $\chi(G)$, for a given graph G .

Among many practical problems that can be modeled using graph coloring, Frequency Assignment problems are perhaps the most famous ones [1]. There are several variations of the VERTEX COLORING PROBLEM that were defined in

order to model the specific constraints of the practical applications related to frequency assignment in networks. The BACKBONE COLORING PROBLEM was defined by Broersma et al. [6,?] in the context of Frequency Assignment Problems where certain channels of communication were more demanding than others.

Formally, given a graph G , a spanning subgraph H of G , called the *backbone* of G , and two positive integers q and k , a q -backbone k -coloring of (G, H) is a k -coloring c of G for which $|c(u) - c(v)| \geq q$, for every $uv \in E(H)$. The q -backbone chromatic number of (G, H) , denoted by $BBC_q(G, H)$, is the minimum k for which there exists a q -backbone k -coloring of (G, H) . The BACKBONE COLORING PROBLEM consists in determining $BBC_q(G, H)$. In this work, we focus on the case $q = 2$ and thus we usually omit q from the notation.

In their seminal article, Broersma et al. observe that

$$BBC(G, H) \leq 2 \cdot \chi(G) - 1. \quad (1)$$

This can be easily seen by considering an optimal coloring of G that uses only odd colors. Note that, thanks to the Four Color Theorem [2,3], whenever G is a planar graph and H is any spanning subgraph of G , we get an upper bound of 7 to the backbone chromatic number of (G, H) . However, when H is a spanning tree of G , Broersma et al. conjecture that this upper bound is in fact 6, and they show that this would be best possible [8].

Conjecture 1 ([8]). If G is a planar graph and T is a spanning tree of G , then

$$BBC(G, T) \leq 6.$$

In the literature, the only result approaching directly this conjecture shows that it holds whenever T has diameter at most 4 [9].

The authors in [10,11,12] consider special backbone k -colorings where the color space is “circular”, i.e., it behaves as \mathbb{Z}/k . More formally, given a graph G , a spanning subgraph H of G , and a positive integer q , a *circular q -backbone k -coloring* of (G, H) is a function $c : V(G) \rightarrow \{1, \dots, k\}$ such that $q \leq |c(u) - c(v)| \leq k - q$, for every $uv \in E(H)$. The *circular q -backbone chromatic number* of (G, H) , denoted by $CBC_q(G, H)$, is the smallest k for which there exists a circular q -backbone k -coloring of (G, H) . Once more, we quite often omit the index q whenever $q = 2$. In order to simplify the notation, we often write CBC- k -coloring instead of circular 2-backbone k -coloring.

Note that any CBC- k -coloring of (G, H) is also a backbone k -coloring of (G, H) , and, conversely, if c is a backbone k -coloring of (G, H) , then it can also be seen as a CBC- $(k + 1)$ -coloring of (G, H) . Therefore we get:

$$BBC(G, H) \leq CBC(G, H) \leq BBC(G, H) + 1. \quad (2)$$

Consequently, as far as Conjecture 1 is not proved to be true, then the following circular version of it is also opened:

Conjecture 2. If G is a planar graph and T is a spanning tree of G , then

$$CBC(G, T) \leq 7.$$

One may observe that a graph G whose chromatic number is $\chi(G) = k$, satisfies $CBC_2(G, H) \leq 2k$, by combining Inequalities 1 and 2. Steinberg conjectures that every planar graph G having no C_4 or C_5 as subgraph satisfies $\chi(G) \leq 3$ [13]. Consequently, one may wonder whether:

Conjecture 3. If G is a planar graph having no C_4 or C_5 as subgraph, then $CBC_2(G, H) \leq 6$, for every backbone $H \subseteq G$.

Notice that Conjecture 3 is in fact equivalent to Steinberg's Conjecture when $H = G$.

In this paper, we prove particular cases of Conjectures 2 and 3.

1.1 Matching Backbones

It is known that if G is a 3-colorable graph and M is a matching of G , then $BBC(G, M) \leq 4$ [7]. Combining this result with Inequality 2, we observe that if Steinberg's Conjecture is true, then $CBC(G, M) \leq 5$, whenever G is a planar graph without cycles of length 4 or 5, and M is a matching of G . We first prove that this bound holds, giving yet more evidence to the validity of Steinberg's Conjecture:

Theorem 1. *If G is a planar graph without cycles of length 4 or 5 as subgraph, and M is a matching of G , then $CBC(G, M) \leq 5$.*

In [7] the authors prove that $BBC(G, M) \leq 6$, whenever G is a planar graph and M is a matching. They also ask whether $BBC(G, M) \leq 5$ holds, and whether $BBC(G, M) \leq 6$ can be proved without using the Four Color Theorem. We partially answer both questions by showing that:

Theorem 2. *If G is a plane graph with no two faces of degree 3 that share an edge, and M is a matching in G , then $CBC(G, M) \leq 6$.*

Although our result restricts the class of graphs when compared to the result presented in [7], it is stronger on this restricted class since we deal with *circular* backbone colorings instead. We mention that our result points to a positive answer to the question about whether $BBC(G, M) \leq 5$, and that our proof does not use the Four Color Theorem.

1.2 Linear Forest Backbones

Finally, we also study more general backbones. A forest is called *linear* if its components are paths.

In [4], the authors investigate $CBC(G, F)$ in the light of Steinberg's Conjecture [13]. Araujo et al. prove that if G is a planar graph with no cycles of length 4 or 5, then $CBC(G, F) \leq 7$ whenever F is a spanning forest of G , and that $CBC(G, F) \leq 6$, whenever F is a spanning linear forest of G [4]. Observe that their results partially solve Conjectures 2 and 3.

The last result we present in this work is similar to theirs by considering planar graphs with no cycles of length 4 and linear forests as backbones.

Theorem 3. *If G is a planar graph without cycles of length 4 as subgraph, and F is a linear spanning forest of G , then $CBC(G, F) \leq 7$.*

Although in our proof we can consider graphs that have C_5 as subgraph, we need an extra color than in the previous result in the literature. However, this was expected since our efforts were done towards an answer to Conjecture 2.

The remainder of this text is organized as follows: in Section 2, we introduce basic notation and results. Then, we prove Theorems 1, 2 and 3 in Sections 3, 4 and 5, respectively.

2 Preliminaries

For the basic definitions about simple graphs and planar graphs, we refer the reader once again to [5].

Given a statement P , and a partially ordered set (\mathcal{S}, \preceq) , we denote by $P(\mathcal{S})$ the set $\{S \in \mathcal{S} \mid P \text{ holds for } S\}$. And we say that $S \in \mathcal{S}$ is a *minimal counterexample for P* if $S \notin P(\mathcal{S})$, and $S' \in P(\mathcal{S})$ for every $S' \in \mathcal{S}$ such that $S' \prec S$. In our proofs, we consider minimal counterexamples to our theorems. For this, we consider a pair (G', H') to be smaller than a pair (G, H) if $G' \subset G$ and $H' \subseteq H$; in this case we say that (G', H') is a *subpair* of (G, H) .

In what follows, given a minimal counterexample (G, H) to one of our theorems, we get a contradiction by being successful in extending a partial $CBC-k$ -coloring of (G', H') to (G, H) , where (G', H') is a subpair of (G, H) . The following lemma presented in [4] will be useful. It can be easily proved by considering a $CBC-k$ -coloring of $(G - u, H - u)$ and observing that it can be extended to a $CBC-k$ -coloring of (G, H) .

Lemma 1 ([4]). *If (G, H) is minimal such that $CBC(G, H) > k$, then, for every $u \in V(G)$, we have that $d_G(u) + 2d_H(u) \geq k$.*

The general technique used to prove the above lemma is also extensively applied in the remainder of the text. Because of this, we introduce the following definitions and notation.

Given a positive integer k , we denote the set $\{1, \dots, k\}$ by $[k]$, and given $c \in [k]$, we denote by $\langle c \rangle$ the set $\{d \in [k] \mid |c - d| \leq 1 \text{ or } |c - d| \geq k - 1\}$ (the colors adjacent to c in the circular space $[k]$). Also, we denote the power set of $[k]$ by $2^{[k]}$. Given a pair (G, H) , a subgraph $G' \subset G$, and a CBC- k -coloring ψ of $(G', H[V(G')])$, we define, for each $u \in V(G) \setminus V(G')$, the set of *available colors* for u in ψ :

$$A_\psi(u) = [k] \setminus (\psi(N_{G'}(u)) \cup \{\langle \psi(v) \rangle \mid v \in N_{H'}(u)\}).$$

Also, we denote $|A_\psi(u)|$ by $a_\psi(u)$.

3 Proof of Theorem 1

In order to prove Theorem 1, we need the following lemma, proved in [4].

Lemma 2 ([4]). *Let G be a plane graph without cycles of length 4 or 5, $G \neq K_3$, and let n and f_3 denote the number of vertices of G and number of faces of degree 3 in G , respectively. Then,*

$$\sum_{v \in V(G)} d(v) \leq 3n + \frac{3f_3}{2} - 6.$$

We use the discharging method to prove that if (G, M) is a minimal counterexample to Theorem 1, then Lemma 2 does not hold for G . This means that no counterexample can exist and that the theorem holds. The following lemma will be useful.

Lemma 3. *Let (G, M) be a minimal counterexample to Theorem 1. Then, we have $\delta(G) \geq 3$. Furthermore, if $u \in V(G)$ has degree 3, then u is incident to some edge in M , say uw , and w is such that $d(w) \geq 5$.*

Proof. Let $u \in V(G)$, and denote by T the subgraph $(V(G), M)$. By Lemma 1 and because $d_T(u) \leq 1$, we get that $d_G(u) \geq 3$. Similarly, if $d_G(u) \leq 4$, we must have $d_T(u) = 1$. So, suppose that $u \in V(G)$ has degree 3 and let $w \in V(G)$ be such that $uw \in M$. By contradiction, suppose that $d(w) \leq 4$, and let ψ be a CBC-5-coloring of $(G - u - w, M - uw)$. Note that $a_\psi(u) \geq 3$ and $a_\psi(w) \geq 2$. Therefore, there exists a color $c \in A_\psi(w) \setminus \langle c \rangle \neq \emptyset$. This implies that ψ can be extended to (G, M) , a contradiction.

Denote by F_3 the set of faces of degree 3 of G . We start by giving charge $d(v) - 3$ for every $v \in V(G)$, and $-\frac{3}{2}$ for every $t \in F_3$. We want to distribute

the charge between the vertices of G and the faces in F_3 in such a way as to ensure that at the end, each vertex and each face in F_3 has nonnegative charge. Because the total amount of charge does not change, we get (below, f_3 and n represent $|F_3|$ and $|V(G)|$, respectively):

$$\sum_{v \in V(G)} (d(v) - 3) - \frac{3f_3}{2} \geq 0 \Leftrightarrow \sum_{v \in V(G)} d(v) \geq 3n + \frac{3f_3}{2}.$$

This contradicts Lemma 2. To prove this can be done, we apply the following discharging rules. Below, given $u \in V(G)$, we denote by $F_3(u)$ the set of faces of degree 3 containing u .

Rule 1 For each $uw \in M$ such that $d(u) = 3$, send $\frac{1}{2}$ charge from w to u .

Rule 2 For each $u \in V(G)$ and each $t \in F_3(u)$, send charge $\frac{1}{2}$ from u to t .

Proof (of Theorem 1). For each $x \in V(G) \cup F_3$, denote by $\mu_0(x), \mu_1(x), \mu_2(x)$ the charge of x before Rule 1 has been applied, before Rule 2 has been applied and after Rule 2 has been applied, respectively. Because M is a matching, no vertex is incident to more than one edge in M . Thus, by Lemma 3, we get the following:

- If $d(u) = 3$, then $\mu_1(u) = \frac{1}{2}$;
- If $d(u) = 4$, then $\mu_1(u) = \mu_0(u) = 1$; and
- If $d(u) \geq 5$, then $\mu_1(u) \geq \mu_0(u) - \frac{1}{2} = \frac{2d(u)-7}{2}$.

Now, for each $u \in V(G)$, denote by $f_3(u)$ the value $|F_3(u)|$. Note that, since G has no cycles of length 4, no two faces in F_3 can share an edge. This implies that $f_3(u) \leq \lfloor \frac{d(u)}{2} \rfloor$. One can verify by what is said above that $\mu_1(u) \geq \frac{d(u)}{4} \geq \frac{f_3(u)}{2}$. This means that after distributing charge $1/2$ to each $t \in F_3(u)$, we get that u still has non-negative charge, i.e., $\mu_2(u) \geq 0$ for every $u \in V(G)$. Finally, because each $t \in F_3$ receives charge $1/2$ from each vertex in t , we get $\mu_2(t) = \mu_0(t) + 3/2 = 0$.

4 Proof of Theorem 2

Consider a plane graph G and its dual G^* , and let F_3 be the set of faces of degree 3 in G (alternatively, the set of vertices of degree 3 in G^*). We denote the graph $G^* - F_3$ by G_4^* , and say that a component of G_4^* is an *island of G* . Also, if H is an acyclic component of G_4^* such that $d_{G^*}(f) = 4$, for every $f \in V(H)$, then we say that H is a *bad island of G* . We denote the set of bad islands of G by Γ and we let γ denote $|\Gamma|$. Let $f \in F_3$ and H be an island of G ; we say that f *share an edge with H* if $N_H(f) \neq \emptyset$ (i.e., if f and f' share an edge in G for some $f' \in V(H)$). Also, we denote by $\Gamma(f)$ the set of bad islands that share an edge with f .

Lemma 4. *Let G be a plane graph with no two faces of degree 3 sharing an edge, and let f_3 denote the number of faces of degree 3 in G . Then,*

$$\sum_{v \in V(G)} d(v) \leq 5|V(G)| + \gamma - f_3 - 10.$$

Proof. Let f_4 denote the number of faces of degree 4 in G , $|E(G)|$ be denoted by m , \mathcal{F} denote the set of faces of G and, given $f \in \mathcal{F}$, let $|f|$ denote the degree of f . We claim that:

$$3f_3 + f_4 \leq m + \gamma \quad (3)$$

This implies that $\sum_{f \in \mathcal{F}} (|f| - 5) \geq -2f_3 - f_4 \geq -m - \gamma + f_3$. On the other hand $\sum_{f \in \mathcal{F}} (|f|) - 5|\mathcal{F}| = 2m - 5|\mathcal{F}|$. Combining these and applying Euler's Formula we get (below, n denotes $|V(G)|$):

$$2m - 5(2 - n + m) \geq -m - \gamma + f_3 \iff 2m \leq 5n + \gamma - f_3 - 10$$

It remains to prove Inequality 3. For this, we partition $E(G)$ in E_3, \overline{E}_3 , where E_3 is described below and $\overline{E}_3 = E(G) \setminus E_3$.

$$E_3 = \{e \in E(G) \mid e \text{ is in the boundary of some face of degree 3}\}.$$

Because G has no two faces of degree 3 sharing an edge, we get $|E_3| = 3f_3$. We prove that $|\overline{E}_3| \geq f_4 - \gamma$, thus finishing the proof. For this, note that if $e \in \overline{E}_3$, then there is an edge e^* in G_4^* related to e . On the other hand, if $e^* \in E(G_4^*)$, then e^* is related to an edge $e \in E(G)$ that separates faces of degree at least 4; hence, $e \in \overline{E}_3$. Therefore, $|\overline{E}_3| = |E(G_4^*)|$. Finally, because the number of edges in any graph is at least the number of vertices minus the number of acyclic components of the graph, we get:

$$|\overline{E}_3| \geq |V(G_4^*)| - \gamma \geq f_4 - \gamma.$$

□

Now, by supposing that there exists a counterexample (G, M) to Theorem 2, we use the discharging method to get a contradiction to Lemma 4. For this, start by giving charge $d(v) - 5$ to each $v \in V(G)$, charge 1 to each $f \in F_3$, and charge -1 to each $b \in \Gamma$. Then, we apply discharging rules and ensure that this initial charge can be redistributed in the graph in such a way that every vertex, every face of degree 3 and every bad island have non-negative charge. We get a contradiction since:

$$\sum_{v \in V(G)} (d(v) - 5) + f_3 - \gamma \geq 0 \iff \sum_{v \in V(G)} d(v) \geq 5n + \gamma - f_3.$$

We need the following lemma.

Lemma 5. *Let (G, M) be a minimal counterexample to Theorem 2. Then, we have $\delta(G) \geq 4$. Furthermore, if $u \in V(G)$ has degree 4, then u is incident to an edge of M , say uw , and $d(w) \geq 6$.*

Proof. Let T denote the subgraph $(V(G), M)$. By Lemma 1, we get $\delta(G) \geq 4$, and that $d_T(u) = 1$ whenever $d_G(u) \leq 5$. So, consider $u \in V(G)$ with degree 4, and suppose that $d(w) \leq 5$, where w is such that $uw \in M$. Let ψ be a CBC-6-coloring of $(G - \{u, w\}, M - \{u, w\})$. Then, $a_\psi(w) \geq 3$ and $a_\psi(u) \geq 2$. Therefore, there exists a color $c \in A_\psi(u)$ such that $A_\psi(w) \setminus \langle c \rangle \neq \emptyset$, which implies that ψ can be extended to (G, M) , a contradiction.

Let V_4 be the set of vertices with degree 4 in G , and for each $u \in V_4$, denote by u^* the vertex such that $uu^* \in M$. The discharging rules are the following:

Rule 1 *For each $f \in F_3$, send charge $\frac{1}{3}$ from f to each $b \in \Gamma(f)$.*

Rule 2 *For each $u \in V_4$, send charge 1 from u^* to u .*

Proof (of Theorem 2). For each $x \in V(G) \cup F_3 \cup \Gamma$, let $\mu_0(x), \mu_1(x), \mu_2(x)$ denote the charge of x before Rule 1, after Rule 1, and after Rule 2 has been applied, respectively. Recall that $\mu_0(v) = d(v) - 5$, for every $v \in V(G)$; $\mu_0(f) = 1$, for every $f \in F_3$; and $\mu_0(b) = -1$, for every $b \in \Gamma$.

Because M is a matching and by Lemma 5, we get that $\mu_2(v) \geq 0$, for every $v \in V(G)$. Also, for each $f \in F_3$, we have $|\Gamma(f)| \leq 3$; hence $\mu_2(f) = \mu_1(f) = \mu_0(f) - |\Gamma(f)|/3 \geq 0$. It remains to prove that each bad island also ends up with non-negative charge. So, consider a bad island of G , i.e., an acyclic component H of G_4^* such that each $f \in V(H)$ has degree 4 in G^* . If $V(H) = \{f\}$, because two faces of degree 3 in G intersect in at most one vertex, we get that f corresponds to an induced cycle of length 4 in G , which implies that f is adjacent to 4 distinct vertices of F_3 . And if $|V(H)| \geq 2$, then H has at least one leaf, say f ; as before, we get that f is adjacent to at least 3 distinct vertices of F_3 . In any case, we get that $y = |\{f \in F_3 \mid H \in \Gamma(f)\}| \geq 3$, which implies that $\mu_2(H) = \mu_1(H) = \mu_0(H) + y/3 \geq 0$.

5 Linear Forest Backbone

We prove Theorem 3 in this section using the same general strategy, except that the structural properties needed are more complex. In the previous sections, a simple lemma concerning at most two vertices, say u and v , was enough to say that a CBC- k -coloring ψ of $(G - u - v, H - u - v)$ could be extended to (G, H) . Here, the backbone is a linear tree and therefore we sometimes need to remove entire subpaths from a minimal counterexample (G, H) . For this, we work with the lists A_ψ in a more clever way. This is done in the next subsection.

5.1 Forbidden Structures

Let (H, P) be such that $P \subseteq H$, k be a positive integer, and $\mathcal{L} : V(H) \rightarrow 2^{[k]}$. If there exists a CBC- k -coloring ψ of (H, P) such that $\psi(v) \in \mathcal{L}(v)$, for all $v \in V(H)$, then we say that (H, P) is \mathcal{L} -CBC- k -colorable. Throughout the proof, we sometimes consider \mathcal{L} to be smallest possible in the context. This is not a problem since whenever (H, P) is \mathcal{L} -CBC- k -colorable and \mathcal{L}' is such that $\mathcal{L}(v) \subseteq \mathcal{L}'(v)$, for every $v \in V(H)$, we also have that (H, P) is \mathcal{L}' -CBC- k -colorable.

Consider a pair (H, P) such that P is a Hamiltonian path of H , and write P as (v_1, \dots, v_n) . Also, let $\mathcal{L} : V(H) \rightarrow 2^{[7]}$ be a list assignment for H , and $\mathcal{L}' : V(H') \rightarrow 2^{[7]}$ be a list assignment for $H' \subseteq H$. We use the reduction rule below to prove the non-existence of certain structures in a minimal counterexample to Theorem 3. We denote the values $|\mathcal{L}(x)|$ and $|\mathcal{L}'(x)|$ by $\ell(x)$ and $\ell'(x)$, respectively.

Reduction Rule: $((H', P'), \mathcal{L}')$ is a *reduction* of $((H, P), \mathcal{L})$ on v_1 if:

- $H' = H - v_1$;
- $P' = P - v_1$;
- $\ell'(v_2) \geq \ell(v_2) - 2$;
- $\ell'(x) \geq \ell(x) - 1$, for every $x \in N(v_1) \setminus \{v_2\}$;
- $\ell'(x) = \ell(x)$, for every $x \in V(H) \setminus N[v_1]$; and
- If $\mathcal{L}(v_2) \setminus \mathcal{L}'(v_2) = \{c, d\}$, then $|\langle c \rangle \cup \langle d \rangle| \leq 5$.

We say that a reduction $((H', P'), \mathcal{L}')$ of $((H, P), \mathcal{L})$ on v_1 is *extendable* if every \mathcal{L}' -CBC-7-coloring of (H', P') can be extended to an \mathcal{L} -CBC-7-coloring of (H, P) . The following lemma gives sufficient conditions for $((H, P), \mathcal{L})$ to have an extendable reduction.

Lemma 6. *Let H be any graph, $P = (v_1, \dots, v_n)$ be a Hamiltonian path of H , and consider $\mathcal{L} : V(H) \rightarrow 2^{[7]}$. If the conditions below hold, then $((H, P), \mathcal{L})$ has an extendable reduction on v_1 .*

1. $d(v_1) \leq 4$;
2. $\ell(v_1) \geq 1 + d(v_1)$; and
3. If $d(v_1) = 4$, and c and d are the colors not in $\mathcal{L}(v_1)$, then $|\langle c \rangle \cup \langle d \rangle| \leq 5$.

Proof. Without loss of generality, suppose that $\ell(v_1) = 1 + d(v_1)$. First, suppose that $d(v_1) = 1$. If $\mathcal{L}(v_1)$ has two consecutive colors, then remove both from $\mathcal{L}(v_2)$; if $\mathcal{L}(v_1) = \{c - 1, c + 1\}$ for some $c \in [7]$, then remove c from $\mathcal{L}(v_2)$; otherwise, do not change $\mathcal{L}(v_2)$. Let \mathcal{L}' be the obtained function. One can see that $((H - v_1, P - v_1), \mathcal{L}')$ is a reduction of $((H, P), \mathcal{L})$ on v_1 . Let ψ be an \mathcal{L}' -CBC-7-coloring of $(H - v_1, P - v_1)$; if no such coloring exists, then the lemma holds by vacuity. By the choice of the removed colors, note that $\mathcal{L}(v_1) \setminus \langle \psi(v_2) \rangle \neq \emptyset$, which means that ψ can be extended to v_1 .

Now, consider $d(v_1) > 1$. First, suppose that there exists $c \in \mathcal{L}(v_1)$ such that $\{c - 1, c + 1\} \cap \mathcal{L}(v_1) = \emptyset$. Let \mathcal{L}' be obtained by removing $c - 1$ and $c + 1$ from

$\mathcal{L}(v_2)$, and c from $\mathcal{L}(x)$, for every $x \in N(v_1) \setminus \{v_2\}$. Then $((H - v_1, P - v_1), \mathcal{L}')$ is a reduction of $((H, P), \mathcal{L})$ on v_1 , and we want to show that it is extendable. So let ψ be an \mathcal{L}' -CBC-7-coloring of $(H - v_1, P - v_1)$, and let $F = \psi(N(v_1)) \cup \langle \psi(v_2) \rangle$, the set of colors that are forbidden for v_1 . If $\psi(v_2) \neq c$, we can color v_1 with c . Otherwise, we get $|\mathcal{L}(v_1) \cap \langle \psi(v_2) \rangle| = 1$, which implies that

$$|\mathcal{L}(v_1) \cap F| \leq |\mathcal{L}(v_1) \cap \psi(N(v_1) \setminus \{v_2\})| + |\mathcal{L}(v_1) \cap \langle \psi(v_2) \rangle| \leq d(v_1) - 1 + 1. \quad (4)$$

Since $\ell(v_1) = d(v_1) + 1$, there is a color in $\mathcal{L}(v_1) \setminus F$ with which we can color v_1 .

Finally, suppose that:

(*) $\{c - 1, c + 1\} \cap \mathcal{L}(v_1) \neq \emptyset$, for every $c \in \mathcal{L}(v_1)$.

Because $2 \leq d(v_1) \leq 4$ and $\ell(v_1) = d(v_1) + 1$, we know that at least two colors are not in $\mathcal{L}(v_1)$ and that $\ell(v_1) \geq 3$. Without loss of generality and by (*), we can suppose that $\{1, 2\} \subset \mathcal{L}(v_1)$ and that $7 \notin \mathcal{L}(v_1)$. We claim that we can also suppose that $6 \notin \mathcal{L}(v_1)$. Suppose otherwise; by (*) we get that $\{1, 2, 5, 6\} \subseteq \mathcal{L}(v_1)$. If $\{3, 4\} \cap \mathcal{L}(v_1) = \emptyset$, then we rotate the colors so that 1 coincides with 5 and the desired property holds. Otherwise, we get a contradiction to Property 3 since $|\langle c \rangle \cup \langle 7 \rangle| = 6$ where $c \in \{3, 4\} \setminus \mathcal{L}(v_1)$. Now, let \mathcal{L}' be obtained by removing 1 from $\mathcal{L}(v_i)$, for every $v_i \in N(v_1) \setminus \{v_2\}$, and 1 and 2 from $\mathcal{L}(v_2)$, and let ψ be an \mathcal{L}' -CBC-7-coloring of $(H - v_1, P - v_1)$. If $\psi(v_2) \neq 7$, we can color v_1 with 1. Otherwise, since $\{6, 7\} \cap \mathcal{L}(v_1) = \emptyset$, we get $|\langle \psi(v_2) \rangle \cap \mathcal{L}(v_1)| = 1$ and, again by Inequality 4, we get that there must exist a color in $\mathcal{L}(v_1)$ with which we can color v_1 .

Now, we want to apply the above lemma to our problem. So, consider a planar graph G with no cycles of length 4, a generating linear forest F of G , and a subpath P of F with certain properties. If (G, F) is a minimal counterexample to Theorem 3, we know that there exists a CBC-7-coloring ψ of $(G - P, F - P)$. We iteratively apply Lemma 6, starting with $((G[V(P)], P), A_\psi)$, until we end up with a single vertex with list of size at least one. This implies that there exists an A_ψ -CBC-7-coloring ψ' of $(G[V(P)], P)$, which in turn implies that ψ can be extended to a CBC-7-coloring of (G, F) , thus contradicting the choice of (G, F) . This ensures the non-existence of such a path P in a minimal counterexample. Before we present the types of paths that cannot occur in a minimal counterexample, we need a further definition.

Let (G, F) be as in the previous paragraph, and T be a component of F . If P is a maximal subpath of T containing only vertices of degree at most 5 in G , we say that P is a *heavy subpath* of T . The next lemma follows easily from Lemma 1 and the fact that F is a linear forest.

Lemma 7. *Let (G, F) be a minimal counterexample to Theorem 3. Then, we have $\delta(G) \geq 3$, and if $v \in V(G)$ is such that $d_G(v) \leq 4$, then $d_F(v) = 2$.*

Lemma 8. *Let (G, F) be a minimal counterexample to Theorem 3, and P be a heavy subpath of a component of F . The following hold.*

- (a) If P has one vertex v of degree 3, then $d(u) = 5, \forall u \in V(P) \setminus \{v\}$;
- (b) If P contains a leaf of F , then $d(u) = 5, \forall u \in V(P)$; and
- (c) P has at most two vertices of degree 4.

Proof. Below, we consider a subpath P' of P , and denote by H the subgraph $G[V(P')]$. We prove that whenever P does not satisfy one of the assertions, then, letting ψ be a CBC-7-coloring of $(G - H, F - H)$, we get that (H, P') is A_ψ -CBC-7-colorable, contradicting the fact that (G, F) is a minimal counterexample to Theorem 3. We recall that, by Lemma 7, we have $\delta(G) \geq 3$ and $d_G(u) \geq 5$ whenever $d_F(u) \leq 1$.

First, suppose that either (a) or (b) does not hold, and let $P' = (v_1, v_2, \dots, v_q)$ be a shortest subpath of P such that $q \geq 2$, $d(v_1) \leq 4$, and either $d(v_q) = 3$ or v_q is a leaf in P . Also, let ψ be a CBC-7-coloring of $(G - H, F - H)$. We construct a sequence R_1, \dots, R_q such that $R_1 = ((H, P'), A_\psi)$; R_i is an extendable reduction of R_{i-1} on v_{i-1} , for each $i \in \{2, \dots, q\}$; and the list available for v_q in R_q , say A_q , is nonempty. Observe that this leads to a contradiction since a coloring of v_q with any $c \in A_q$ can be extended to an A_ψ -CBC-7-coloring of (H, P') by the definition of extendable reduction. For each $i \in \{1, \dots, q\}$ we write R_i as $((H_i, P_i), A_i)$. Observe that $P_i = (v_i, \dots, v_q)$ and that $H_i = H[\{v_i, \dots, v_q\}]$, and denote by $\ell_i(v)$ the value $|A_i(v)|$, for each $v \in \{v_i, \dots, v_q\}$. In order to obtain the desired sequence of extendable reductions, we want to apply Lemma 6. For this, we need to ensure that, at the beginning and after each step i of the procedure, the inequalities below hold.

$$\ell_i(v_j) \geq d_{H_i}(v_j) + 2, \text{ for every } j \text{ such that } i < j < q. \quad (5)$$

$$\ell_i(v_i) \geq d_{H_i}(v_i) + 1, \text{ if } i < q. \quad (6)$$

$$\ell_i(v_q) \geq \begin{cases} d_{H_i}(v_q) + 2, & \text{if } i < q \\ 1, & \text{otherwise} \end{cases} \quad (7)$$

Claim. If Inequalities (5), (6), and (7) hold for R_i , with $1 \leq i < q$, then R_i has an extendable reduction on v_i .

Proof: Because $d(v_j) \leq 5$, for every $v_j \in V(P')$, and by Inequality (6), we get that Conditions (1) and (2) of Lemma 6 hold. Now, suppose that $d_{H_i}(v_i) = 4$. Recall that $d_G(v_1) \leq 4$; hence $1 < i < q$, which implies that $d_H(v_i) = 5$. But since $d_{H_i}(v_i) = 4$, this means that $N_G(v_i) = N_{H_i}(v_i) \cup \{v_{i-1}\}$, which implies that $A_{i-1}(v_i) = [7]$. Then, Condition (3) follows by the definition of reduction. ■

We first argument that these inequalities initially hold. Recall that $H_1 = H$, $P_1 = P'$, and $A_1 = A_\psi$. First, consider any $j \in \{2, \dots, q-1\}$. Since F is a linear tree, we have that $N_F(v_j) \subseteq P'$, which means that $\ell_1(v_j) \geq 7 -$

$d_{G-H}(v_j) = 7 - (d_G(v_j) - d_H(v_j))$. By the choice of v_1 and v_q , we know that $d_G(v_j) = 5$, which in turn implies Inequality (5). Now, by Lemma 7, we know that $d_P(v_1) = 2$; so let $v \in N_P(v_1) \setminus \{v_2\}$. Note that v forbids 3 colors for v_1 , while each other colored neighbor of v_1 forbids just one color. This gives us that $\ell_1(v_1) \geq 7 - (d_{G-H}(v_1) + 2d_{P-P'}(v_1)) = 5 - (d_G(v_1) - d_H(v_1)) \geq d_H(v_1) + 1$. Analogously, for v_q we get: if $d(v_q) = 3$, then $d_F(v_q) = 2$ and $\ell_1(v_q) \geq d_H(v_q) + 2$; and if v_q is a leaf in P , then by Lemma 7 we get $d_G(v_q) = 5$, and as before $\ell_1(v_q) = 7 - d_{G-H}(v_q) \geq d_H(v_q) + 2$.

Now, suppose that we are at step i of our construction, $1 \leq i < q$, and let R_{i+1} be an extendable reduction of R_i . We want to prove that Inequalities (5), (6), and (7) also hold for R_{i+1} . First, note that if $v_j \in N(v_i) \setminus \{v_{i+1}\}$, then both $d_{H_{i+1}}(v_j)$ and $\ell_{i+1}(v_j)$ decrease by exactly 1; hence, Inequality (5) holds, as well as Inequality (7) in the case where $i < q - 1$. Similarly, $d_{H_{i+1}}(v_{i+1})$ decreases by 1, while $\ell_{i+1}(v_{i+1})$ decreases by at most 2; hence, if $i < q - 1$, we have that $\ell_i(v_{i+1}) \geq d_{H_i}(v_{i+1}) + 2$, which means that Inequality (6) also holds for R_{i+1} . Finally, suppose that $i = q - 1$. Then $\ell_{q-1}(v_q) \geq d_{H_{q-1}}(v_q) + 2 = 3$, and by the definition of reduction we get that $\ell_q(v_q) \geq 1$, i.e., Inequality (7) holds also when $i = q - 1$, and we are done proving (a) and (b).

Finally, in order to prove (c), suppose that $d(v) \geq 4$, for every $v \in V(P)$, and let $u, v, w \in V(P)$ be the closest three vertices of degree 4 in P , where v is between u and w . Write the subpath of P between u and w as $P' = (v_1 = u, v_2, \dots, v_q = w)$ and let $v_p = v$. Denote $G[V(P')]$ by H , and let ψ be a CBC-7-coloring of $(G - H, F - H)$. Note that:

- For each $z \in V(P') \setminus \{u, v, w\}$, we get $a_\psi(z) \geq 7 - d_{G-H}(z) = 7 - (d_G(z) - d_H(z)) = 2 + d_H(z)$;
- For $z \in \{u, w\}$, we get $a_\psi(z) \geq 4 - (d_{G-H}(z) - 1) = 1 + d_H(z)$; and
- $a_\psi(v) = 7 - d_{G-H}(v) = 3 + d_H(v)$.

By arguments similar to the ones made for the first two cases, one can verify that a series of extendable reductions can be made on P' , from v_1 up to v_{p-1} , and from v_q down to v_{p+1} , until we end up with just v_p with non-empty list.

5.2 Discharging Method

In this section, we finish the proof of Theorem 3. For this, we use a definition similar to the one used in the proof of Theorem 2. We make an abuse of language and use the same nomenclature. Consider a plane graph G and its dual G^* , and let F_3 be the set of faces of degree 3 in G (alternatively, the set of vertices of degree 3 in G^*). We denote the graph $G^* - F_3$ by G_5^* , and say that a component of G_5^* is an *island* of G . Also, if H is an acyclic component of G_5^* such that $d_{G^*}(f) = 5$, for every $f \in V(H)$, then we say that H is a *bad island* of G . We denote the set of bad islands of G by Γ and we let γ denote $|\Gamma|$. Also, for $v \in V(G)$, we denote by $\Gamma(v)$ the set of bad islands containing v , and by $\gamma(v)$

the value $|\Gamma(v)|$. If $X \subseteq V(G)$, then $\Gamma(X) = \bigcup_{x \in X} \Gamma(x)$, and $\gamma(X) = |\Gamma(X)|$. In the remainder of the text, although we refer to G as being planar, we are implicitly considering a planar embedding of G and its islands.

Lemma 9. *Let G be a planar graph without cycles of length 4 as subgraph. Then,*

$$|E(G)| \leq 2|V(G)| - 4 + \frac{\gamma}{3}.$$

Proof. Let f_3, f_5 denote the number of faces of degree 3 and 5, respectively, and let $|E(G)|$ be denoted by m . Also, denote by \mathcal{F} the set of faces of G and by $|f|$ the degree of a face $f \in \mathcal{F}$. We claim that:

$$3f_3 + f_5 \leq m + \gamma \quad (8)$$

This implies that $t = \sum_{f \in \mathcal{F}} (|f| - 6) \geq -3f_3 - f_5 \geq -m - \gamma$. On the other hand $t = \sum_{f \in \mathcal{F}} (|f| - 6)|\mathcal{F}| = 2m - 6|\mathcal{F}|$. Combining these and applying Euler's Formula we get (below, n denotes $|V(G)|$):

$$2m - 6(2 - n + m) \geq -m - \gamma \iff m \leq 2n - 4 + \frac{\gamma}{3}$$

It remains to prove Inequality 8. For this, we partition $E(G)$ in E_3, \overline{E}_3 , where E_3 is described below and $\overline{E}_3 = E(G) \setminus E_3$.

$$E_3 = \{e \in E(G) \mid e \text{ is contained in some face of degree 3}\}.$$

Because G has no cycle of length 4, we trivially get that $|E_3| = 3f_3$. We prove that $|\overline{E}_3| \geq f_5 - \gamma$, thus finishing the proof. For this, note that if $e \in \overline{E}_3$, then there is an edge e^* in G_5^* related to e . On the other hand, if $e^* \in E(G_5^*)$, then e^* is related to an edge $e \in E(G)$ that separates faces of degree at least 5; hence, $e \in \overline{E}_3$. Therefore, $|\overline{E}_3| = |E(G_5^*)|$. Finally, because the number of edges in any graph is at least the number of vertices minus the number of acyclic components of the graph, we get:

$$|\overline{E}_3| \geq |V(G_5^*)| - \gamma \geq f_5 - \gamma.$$

Supposing that (G, F) is a minimal counterexample to Theorem 3, we apply the discharging method to prove that $\sum_{v \in V(G)} d(v) \geq 4|V(G)| + \frac{2\gamma}{3}$, contradicting Lemma 9. For this, we start by giving charge $d(v) - 4$ to every $v \in V(G)$, and charge $-2/3$ to every bad island. The discharging rules ensure that every vertex and every bad island end up with a non-negative charge (i.e., Property 1 below holds), which clearly contradicts Lemma 9. The rules are applied in the order they are presented. Also, given $x \in V(G) \cup \Gamma$, the initial charge of x is denoted by $\mu_0(x)$, and the charge of x after Rule i is applied is denoted by $\mu_i(x)$, for each $i \in \{1, \dots, 5\}$.

Property 1. After Rule i is applied, we have that $\mu_i(v) \geq 0$ and $\mu_i(b) \geq 0$, for every vertex v iterated in Rule i and every bad island b containing v .

The proof following each rule is a proof that Property 1 holds after the corresponding rule has been applied.

Rule 1 For every $v \in V(G)$ with $d(v) \geq 6$, send $2/3$ from v to each $b \in \Gamma(v)$.

Proof. Consider $v \in V(G)$ with $d(v) \geq 6$. Because every island containing v receives $2/3$, we just need to prove that $\mu_1(v) \geq 0$. Because G has no cycles of length 4, observe that $\gamma(v) \leq \frac{d(v)}{2}$. This gives us that:

$$\mu_1(v) \geq d(v) - 4 - \frac{2}{3}\gamma(v) \geq d(v) - 4 - \frac{2}{3} \cdot \frac{d(v)}{2} \geq \frac{2}{3}d(v) - 4 \geq 0. \quad (9)$$

The following proposition will be useful in the remainder of the text. Observe that it holds because at least one face containing uv cannot be a face of degree 3, as otherwise we get a cycle of length 4.

Proposition 1. If G is a graph without cycles of length 4, and $uv \in E(G)$, then there exists a face of degree greater than 3 containing uv .

Rule 2 Let $P = (v_1, \dots, v_q)$ be a heavy subpath containing no vertex with degree smaller than 5. We have the following cases:

- R2.1 If P is a component of F , send charge $2/3$ from $\mu_1(v_1) + \mu_1(v_2)$ to every $b \in \Gamma(\{v_1, v_2\})$. After this, if $q \geq 3$, then for each $i \in \{3, \dots, q\}$, send charge $2/3$ from v_i to $b \in \Gamma(v_i) \setminus \Gamma(v_{i-1})$.
- R2.2 Otherwise, let $v_0 \in N_F(v_1) \setminus \{v_2\}$. For every $i \in \{1, \dots, q\}$, send charge $2/3$ from v_i to $b \in \Gamma(v_i) \setminus \Gamma(v_{i-1})$.

Proof. First, note that $\mu_1(v_i) = 1$, for every $i \in \{1, \dots, q\}$. Suppose that P is a component of F . Note that Lemma 1 implies that $q \geq 2$. By Proposition 1, we get that $\gamma(\{v_1, v_2\}) \leq 3$, and that, when $q \geq 3$, then for every $i \in \{3, \dots, q\}$ we get $|\Gamma(v_i) \setminus \Gamma(v_{i-1})| \leq 1$. Property 1 follows.

Now, suppose that P is not a component of G , in which case we can suppose, without loss of generality, that v_0 exists. By the definition of heavy path, we know that $d(v_0) \geq 6$, which, by Rule 1, implies that the island in $\Gamma(v_0) \cap \Gamma(v_1)$ has non-negative charge. Now, applying Proposition 1, for each $v_i \in V(P)$ we get that $|\Gamma(v_i) \setminus \Gamma(v_{i-1})| \leq 1$. Hence, Property 1 follows.

Rule 3 Let $P = (v_1, \dots, v_q)$ be a heavy subpath containing exactly one vertex with degree smaller than 5, namely v_p , and let $v_0 \in N_F(v_1) \setminus P$ and $v_{q+1} \in N_F(v_q) \setminus P$. We have the following cases.

- R3.1 If $q \geq 2$, we can suppose that $p < q$.
 - (i) Send charge $2/3$ from v_i to $b \in \Gamma(v_i) \setminus \Gamma(v_{i-1})$, for each $i \in \{1, \dots, p-1\}$;
 - (ii) Send charge $2/3$ from v_i to $b \in \Gamma(v_i) \setminus \Gamma(v_{i+1})$, for each $i \in \{p+2, \dots, q\}$;

(iii) If $d(v_p) = 3$, then v_{p+1} sends charge 1 to v_p . Otherwise, v_{p+1} sends charge $2/3$ to $b \in \Gamma(v_p) \cap \Gamma(v_{p+1})$.

R3.2 If $q = 1$ and $d(v_1) = 3$, let $b \in \Gamma(v_1)$. Send charge 1 from $\mu_2(v_0) + \mu_2(v_2) + \mu_2(b)$ to v_1 .

Proof. By Lemma 8, we know that v_0 and v_{q+1} exist, and, by Rule 1, we know that the islands in $\Gamma(v_0) \cap \Gamma(v_1)$ and $\Gamma(v_q) \cap \Gamma(v_{q+1})$ have non-negative charge. First, suppose that $q \geq 2$. By arguments similar to the ones in the previous demonstrations, one can see that the vertices in $\{v_1, \dots, v_{p-1}, v_{p+2}, \dots, v_q\}$, as well as the islands containing them, have non-negative charge. Also, note that, by Proposition 1, either $d(v_p) = 3$ and the only island containing v_p also contains v_{p-1} and v_{p+1} , or $d(v_p) = 4$ and the island in $\Gamma(v_p) \cap \Gamma(v_{p+1})$ is the only one that might not be satisfied yet. In either case, one can verify that the rule satisfies v_p or the referred island, depending on the case.

Now, suppose that $q = p = 1$. If $d(v_1) = 4$, then $\Gamma(v_1) \subseteq \Gamma(v_0) \cup \Gamma(v_2)$ and nothing needs to be done; so suppose otherwise. First note that, because $d(v_1) = 3$, the island $b \in \Gamma(v_1)$ also contains v_0 and v_2 . This means that b has received charge from both v_0 and v_2 when Rule 1 is applied; hence $\mu_2(b) = 2/3$. We end the proof by showing that $\mu_2(v_2) = \mu_1(v_2) \geq 2/3$. Note that, since $d(v_1) = 3$ and because G has no cycle of length 4, we can suppose that v_1 has no common neighbor with v_2 . Therefore, if $d(v_2) = 6$, then $\gamma(v_2) = 2$, and applying the first part of Inequality 9, we get that $\mu_2(v_2) = 6 - 4 - 4/3 = 2/3$. On the other hand, if $d(v_2) \geq 7$, we get $\mu_2(v_2) \geq 2/3$ by Inequality 9.

In the next discharging rule, given $X \subseteq V(G)$, we denote $\sum_{v \in X} \mu_3(x)$ by $\mu_3(X)$.

Rule 4 Let $P = (v_1, \dots, v_\ell)$ be a heavy subpath containing exactly two vertices with degree smaller than 5, namely v_p and v_q , $p < q$. Let $v_0 \in N_F(v_1) \setminus P$ and $v_{\ell+1} \in N_F(v_\ell) \setminus P$. Define

$$\beta = \Gamma(V(P)) \setminus \Gamma(\{v_0, v_{\ell+1}\}), \text{ and}$$

$$\mu = \mu_3(V(P)) + \frac{2}{3}|\Gamma(v_0) \cap \Gamma(v_{\ell+1})|.$$

If $\mu \geq \frac{2}{3}|\beta|$, then send $2/3$ from $V(P)$ and $\Gamma(v_0) \cap \Gamma(v_{\ell+1})$ to each $b \in \beta$.

By the condition under which it is applied, Rule 4 clearly satisfies Property 1. However, we still need a final rule for the paths on which the condition $\mu \geq \frac{2}{3}|\beta|$ does not hold. Before we present the rule, we give sufficient conditions for Rule 4 to be applied.

Lemma 10. If P is a heavy subpath containing exactly two vertices with degree smaller than 5, and either $|V(P)| \geq 4$, or $\gamma(V(P)) \leq |V(P)|$, then $\mu \geq \frac{2}{3}|\beta|$.

Proof. Consider $P, v_p, v_q, v_0, v_{\ell+1}, \beta, \mu$ be all defined as in Rule 4 (recall that $v_0, v_{\ell+1}$ exist by Lemma 8). First note that

$$|\beta| = \gamma(V(P)) - |\Gamma(V(P)) \cap \Gamma(\{v_0, v_{\ell+1}\})|.$$

Also, by Proposition 1, we have

$$\gamma(V(P)) \leq 2\ell - (\ell - 1) = \ell + 1.$$

Finally, by Lemma 8, we get that $d(v_p) = d(v_q) = 4$, and $d(v_i) = 5$, for every $v_i \in V(P) \setminus \{v_p, v_q\}$. Hence

$$\mu_3(V(P)) = \ell - 2.$$

Now, denote by t the value $|\Gamma(V(P)) \cap \Gamma(\{v_0, v_{\ell+1}\})|$. By Proposition 1, we know that $t \geq 1$. We analyse the following cases:

- If $t = 1$, then the islands in $\Gamma(v_0) \cap \Gamma(v_1)$ and $\Gamma(v_\ell) \cap \Gamma(v_{\ell+1})$ must be the same, i.e., $\Gamma(v_0) \cap \Gamma(v_{\ell+1}) \neq \emptyset$, and $|\beta| = \gamma(V(P)) - 1$. Therefore,

$$\mu \geq \mu_3(V(P)) + \frac{2}{3} = \ell - 2 + \frac{2}{3} = \ell - \frac{4}{3}.$$

If $\ell \geq 4$, then $|\beta| \leq \ell$ and $\mu \geq \ell - \frac{4}{3} \geq \frac{2}{3}\ell \geq \frac{2}{3}|\beta|$. And if $\gamma(V(P)) \leq \ell$, then $|\beta| \leq \ell - 1$, and, since $\ell \geq 2$, we get $\mu = \ell - \frac{4}{3} \geq \frac{2}{3}(\ell - 1) \geq \frac{2}{3}|\beta|$.

- Now, if $t \geq 2$ and $\ell \geq 4$, then $|\beta| \leq \ell - 1$, and $\mu \geq \ell - 2 \geq \frac{2}{3}(\ell - 1) \geq \frac{2}{3}|\beta|$. Finally, if $t \geq 2$ and $\gamma(V(P)) \leq \ell$, then $|\beta| \leq \ell - 2$ and clearly $\mu \geq \ell - 2 \geq |\beta| \geq \frac{2}{3}|\beta|$.

Now, consider P as in Rule 4 and suppose that the rule is not applied, which means that there might still exist some bad island intersecting $V(P)$ with negative charge. If such an island exists, we call such a path *defective*. Before we present the last discharging rule, we need the lemmas below. We mention that by Lemma 10, if P is defective then $\ell \leq 3$ and $\gamma(V(P)) \geq \ell + 1$, where $\ell = |V(P)|$.

Lemma 11. *Let P be a defective path of size ℓ with extremities v_1 and v_ℓ , and denote by v_2 the neighbor of v_1 in P (hence, it might happen that $\ell = 2$). Also, let $v_0 \in N_F(v_1) \setminus \{v_2\}$, and $v_{\ell+1} \in N_F(v_\ell) \setminus \{v_{\ell-1}\}$. Then, for each $i \in \{1, 2, \ell\}$, we have that v_i is contained in exactly two bad islands (which means that v_i is contained in two 3-faces that separate these bad islands), and $v_{i-1}v_{i+1} \notin E(G)$.*

Proof. First, suppose that $i \in \{1, 2, \ell\}$ is such that v_i is contained in at most one triangle, which means that $\gamma(v_i) \leq 1$. Note that if $\ell = 3$, then $|\Gamma(v_1) \cap \Gamma(v_2) \cap \Gamma(v_3)| - |\Gamma(v_1) \cap \Gamma(v_3)| \leq 0$. This justifies the second line in the equation below.

$$\begin{aligned} \gamma(V(P)) &= |\bigcup_{v_j \in V(P)} \Gamma(v_j)| \\ &\leq \sum_{j \in \{1, 2, \ell\}} \gamma(v_j) - \sum_{j \in \{1, \ell-1\}} |\Gamma(v_j) \cap \Gamma(v_{j+1})| \\ &\leq \sum_{j \in \{1, 2, \ell\} \setminus \{i\}} \gamma(v_j) + \gamma(v_i) - (\ell - 1) \\ &\leq 2(\ell - 1) + 1 - \ell + 1 = \ell \end{aligned}$$

This means that P satisfies Lemma 10, a contradiction. Note also that this actually implies that each v_i is contained in exactly two bad islands.

Now, suppose that $i \in \{1, 2, \ell\}$ is such that $v_{i-1}v_{i+1} \in E(G)$. Note that if $\ell = 3$ and $i = 2$, then $\gamma(V(P)) = \gamma(\{v_1, v_3\})$, and the island in $\Gamma(v_0) \cap \Gamma(v_1)$ also contains v_3 . This implies that $\gamma(V(P)) = 3$, contradicting Lemma 10. So suppose, without loss of generality, that $i = 1$ and let b be the island containing v_0v_2 . Note that $\Gamma(v_1) \subseteq \Gamma(\{v_0, v_2\})$; therefore, $\beta = \Gamma(\{v_2, v_\ell\}) \setminus \Gamma(\{v_0, v_{\ell+1}\})$. First consider $\ell = 2$. If b also contains v_3 , then $|\beta| \leq |\Gamma(v_2) \setminus \{b\}| = 1$, and $\Gamma(v_0) \cap \Gamma(v_3) \neq \emptyset$, which implies $\mu \geq \frac{2}{3}|\beta|$. And if b does not contain v_3 , then $\Gamma(v_2) \subseteq \Gamma(\{v_0, v_3\})$, in which case $\beta = \emptyset$. Both cases are contradictions. Therefore, suppose that $\ell = 3$, and let B denote $\Gamma(\{v_0, v_4\})$. Note that:

$$\begin{aligned} |\beta| &= |\Gamma(\{v_2, v_3\}) \setminus B| \\ &= |(\Gamma(v_2) \setminus B) \cup (\Gamma(v_3) \setminus B)| \\ &\leq |\Gamma(v_2) \setminus B| + |\Gamma(v_3) \setminus B| \leq 2. \end{aligned}$$

The last part holds since $b \in \Gamma(v_2) \cap B$, and $\Gamma(v_3) \cap \Gamma(v_4) \neq \emptyset$ (Proposition 1). If $|\beta| \leq 1$ we are done since $\mu \geq 1$. Therefore, suppose $|\beta| = 2$, in which case we must have $(\Gamma(v_2) \setminus B) \cap (\Gamma(v_3) \setminus B) = \emptyset$. So, let $b_i \in \Gamma(v_i) \setminus B$, for $i = 2$ and $i = 3$, and let $b^* \in \Gamma(v_3) \cap \Gamma(v_4)$. Because $\Gamma(v_2) \cap \Gamma(v_3) \neq \emptyset$ and $b_2 \neq b_3$, we get $b = b^*$, i.e., $b \in \Gamma(v_0) \cap \Gamma(v_4)$. Therefore, we get $\mu \geq 1 + \frac{2}{3} > \frac{4}{3} = \frac{2}{3}|\beta|$, a contradiction.

The next lemma is the final step before we can present the last discharging rule. We denote by Θ the set of bad islands with negative charge, and by D the set of vertices of degree 5 which are contained in some island in Θ .

Lemma 12. *Let $b \in \Theta$, and f be a face of degree 5 in b . Then f contains at least one vertex of D and, if it contains exactly one such vertex, namely u , then b is the only island in Θ that contains u .*

Proof. Let $f = (v_1, \dots, v_5)$ be such that v_i is contained in some defective path, for each $i \in \{1, \dots, 5\}$. Without loss of generality, suppose that $d(v_i) = 4$, for every $i \in \{1, \dots, 4\}$. First, we want to prove that (v_1, \dots, v_5) is an induced cycle in G . So suppose that $v_1v_3 \in E(G)$. Since f is a 5-face in G , we must have that the edge v_1v_3 is traced in the outer side of f . Because $\delta(G) \geq 3$, one can verify that this implies that (v_1, v_2, v_3) is not a 3-face in G , which in turn implies that v_1 is contained in at most one bad island, contradicting Lemma 11. Observe that the same argument can be applied to conclude that $v_iv_j \notin E(G)$, for every $i \in \{1, \dots, 4\}$ and every $j \in \{1, \dots, 5\} \setminus \{i\}$. Now observe that, by Lemma 11, there must exist u_1, \dots, u_5 , where $u_5 \in N(v_1) \cap N(v_5)$, and $u_i \in N(v_i) \cap N(v_{i+1})$, for each $i \in \{1, \dots, 4\}$. This means that every island in Θ is a face of degree 5. We claim that $d(v_5) = 5$. Supposing it holds, let $w \in N(v_5) \setminus \{v_1, v_4, u_4, u_5\}$; also let f_1 be the face containing u_4v_5 different from (v_4, v_5, u_5) , and f_2 be the face containing u_5v_5 different from (u_5, v_5, v_1) . Because G has no cycles of length 4, we know that f_1 and f_2 have degree bigger than 3, and that they share the edge

v_5w . This means that f_1 and f_2 are within the same island t , which implies that $t \notin \Theta$, and the lemma follows, i.e., b is the only island in Θ containing u . It remains to prove our claim.

Suppose by contradiction that $d(v_5) = 4$, and let H denote the induced subgraph $G[\{v_1, \dots, v_5, u_1, \dots, u_5\}]$. Because $d_F(v_i) = 2$ and $N(v_i) \subseteq V(H)$, for every $i \in \{1, \dots, 5\}$, we know that H must contain every edge in F incident to $\{v_1, \dots, v_5\}$. For each v_i , let E_i denote the set $\{uv_i \in E(F)\}$; we know that $|E_i| = 2$. Therefore, if $E_i \cap E_j = \emptyset$, for every $i, j \in \{1, \dots, 5\}$, $i \neq j$, then $|E(H) \cap E(F)| = |\bigcup_{i=1}^5 E_i| = \sum_{i=1}^5 |E_i| = 10 = |V(H)|$, contradicting the fact that F is acyclic. We can then suppose, without loss of generality, that $v_1v_2 \in E(F)$. By Lemmas 8 and 11, we get that $\{u_5v_1, u_2v_2\} \subseteq E(F)$. Also, by Lemma 11, we get $|\{v_3v_4, v_3u_3\} \cap E(F)| \leq 1$ and $|\{v_4v_5, u_4v_5\} \cap E(F)| \leq 1$. This implies that $\{u_5v_5, u_2v_3\} \subseteq E(F)$. It is easy to verify that no matter the choice of edges in E_4 , we get a cycle in F , a contradiction.

The lemma above implies the correctness of our final discharging rule.

Rule 5 Let $K = (D, E)$ be such that $uv \in E$ if and only if u and v are within the same bad island $b \in \Theta$. For each component K' of K , apply one of the following:

- R5.1* If $|V(K')| \geq 2$, let T be a spanning tree of K' and let $uv \in E(T)$. Send charge $2/3$ from $\{u, v\}$ to each island in $\Gamma(\{u, v\})$, and for every $w \in V(T) \setminus \{u, v\}$, send charge $2/3$ from w to the island in $\Gamma(w) \setminus \Gamma(w')$, where $w' \in N_T(w)$ separates w from uv .
- R5.2* If $V(K') = \{u\}$, send $2/3$ from u to the bad island in Θ containing u .

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